Algorithmic information for intermittent systems with an indifferent fixed point

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Abstract

Measuring the average information that is necessary to describe the behaviour of a dynamical system leads to a generalization of the Kolmogorov-Sinai entropy. This is particularly interesting when the system has null entropy and the information increases less than linearly with respect to time. We consider two classes of maps of the interval with an indifferent fixed point at the origin and an infinite natural invariant measure. We calculate that the average information that is necessary to describe the behaviour of its orbits increases with time n approximately as n^α, where $\alpha < 1$ depends only on the asymptotic behaviour of the map near the origin.

1 Introduction

The complexity and unpredictability of a chaotic system has been measured using many different indicators. Among all one of the most important is the Kolmogorov-Sinai (K-S) entropy. Being based on the Shannon's notion of information, it is an average measure of the quantity of information that is necessary to describe each step of the behaviour of the system (with an arbitrary accuracy given by the choice of a partition).

More recently other notions of information content, such as the Kolmogorov-Chaitin *Algorithmic Information Content*, have been applied to dynamical systems. These notions are pointwise and allow to consider the complexity of the behaviour of a single orbit. Hence it is possible to define

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the information $I(x, n, Z)$ contained in n steps of the orbit of a point x with respect to a partition Z of the phase space. This can be done by associating to the orbit of x the symbolic orbit with respect to Z and considering the information content of this string (see Section 2). The average of this pointwise information over an invariant measure μ is strictly related to the K-S entropy $h_\mu(T, Z)$ of the measure μ relative to the partition Z. Indeed for a "typical" point x it holds $I(x, n, Z) \sim n \, h_\mu(T, Z)$ (see Theorem 2.3). When the entropy is null the previous relation becomes $I(x, n, Z) = o(n)$. The many possible different sublinear asymptotic behaviours of $I(x, n, Z)$ correspond to different kinds of "weakly" chaotic dynamics. The importance of this indicator of weak chaos is also confirmed by the relations that have been proved, even in the null entropy case, between the behaviour of the information and many important features of the dynamics, such as sensitivity to initial conditions, dimensions, recurrence ([8],[24],[7],[12],[16]) and global topological complexity indicators ([13], see also [2] for relations between the topological complexity and other physically important features of dynamics).

A class of systems which have a sublinear increase of the information are the systems with an infinite invariant measure (see Theorem 2.4). An important subclass of these consists of maps with an indifferent fixed point, being an important example the map on the interval $[0, 1]$ given by

$$x \mapsto x + x^z \pmod 1 \ \text{ for } z \geq 2 \tag{1}$$

In the above family of maps the origin is a neutrally unstable fixed point, hence an orbit that is sent near the origin can be trapped near the origin for long times. The resulting behaviour is an alternation of chaotic (when the orbit stays far from the origin the map is similar to the baker's map) and regular (when the orbit is trapped near the origin) phases. The expected trapping times can be modulated by varying z. For these reasons this map was introduced in the physical literature in [19] as a model of intermittent turbulent behaviour in fluid dynamics, and the particular statistical properties of the orbits of these maps were used in different fields to model intermittent phenomena (see for example [3]).

From the information point of view these maps exhibit a behaviour that is between the fully chaotic (positive entropy) and the regular one (the dynamics is predictable, the information needed to describe it increases with time n at most as $\log(n)$). This was first discovered in [14] in a piecewise linear example (see Section 4). Their seminal, short paper however does not present a complete mathematical proof of this fact (a lower bound, like

Theorem 3.8 is not proved). Some further study was made in [12] (where the lower bound was proved under some assumption on z) and in [6].

This paper considers both C^1 and piecewise linear (PL) classes of interval maps with an indifferent fixed point. The C^1 case is studied using techniques that are different from the techniques used in the previous literature.

The main result of this paper implies that, in mean with respect to any absolutely continuous probability measure the information of the "Manneville-Pomeau like"(see Definition 3.1) class of maps, for which equation (1) is an example, behaves for $z > 2$ like

$$\mathbb{E}[I(x, n, Z)] \sim n^{\frac{1}{z-1}}$$

that is as a power law with exponent less than 1 (see Theorem 3.9). This shows in particular that the behaviour of the information content for these maps depends only on the local behaviour of the map near the neutrally unstable fixed point at the origin. Some results about the pointwise behaviour of $I(x, n, Z)$ are also given (see Proposition 3.11).

Moreover we study a class of piecewise linear maps, extending the results of [14], finding in particular different behaviours for the mean of the information content (see Section 4).

All our results are based on the definitions of Section 2, where we introduce in an informal way the *Algorithmic Information Content* of a string and the few related facts about algorithmic information theory that we need in the following. Then using these concepts we define the *local* and *global chaos indexes*, which measure the local and average power law behaviour of the information $I(x, n, Z)$ (roughly speaking when $I(x, n, Z) \sim n^\alpha$ then the index is α). The invariance properties of these indexes and the results on the Manneville-Pomeau maps imply, as a simple corollary, that two Manneville-Pomeau maps with different parameter z cannot be absolutely continuously conjugate (Corollary 3.10).

2 Information measures and chaotic dynamical systems

The method we use to study a chaotic dynamical system is based on the idea of a measure of the information contained in the orbits of the system.

Let $\mathcal{A}$ be a finite alphabet and let $\mathcal{A}^n$ be the set of all strings of length n written with letters from $\mathcal{A}$. Then by $\mathcal{A}^*$ we denote the set of finite strings

of any length, that is

$$\mathcal{A}^* = \bigcup_{n=0}^{\infty} \mathcal{A}^n$$

Given a string $s \in \mathcal{A}^*$, the intuitive idea of information contained in s is the length of the smallest binary message from which it is possible to reconstruct s. Thus, formally, the information I is a function $I : \mathcal{A}^* \to \mathbb{N}$.

One of the most important measures for the information content is the *Algorithmic Information Content (AIC)*. In order to define it, it is necessary to define the notion of partial recursive function. We limit ourselves to give an intuitive idea which is very close to the formal definition. We can consider a partial recursive function as a computer C which takes a program p (namely a binary string) as an input, performs some computations, and gives a string $s = C(p)$, written on the given alphabet $\mathcal{A}$, as an output. The AIC of a string s is defined as the shortest binary program p which gives s as its output, namely

$$AIC(s, C) = \min\{|p| : C(p) = s\} \qquad (2)$$

where $|\cdot|$ denotes the binary length of the program p. Up to now the Algorithmic Information Content depends on C, but there is a class of computing machines that allows a definition of information content independent on the particular computer up to a constant. We require that our computer is a universal computing machine. Roughly speaking, a computing machine is called *universal* if it can simulate any other machine if appropriately programmed. That is, C is universal if for each other computer D there is a program p_{CD} such that for each program p it holds $D(p) = C(p_{CD}p)$. In particular the computers we use every day are universal computing machines, provided that we assume that they have virtually infinite memory. For a precise definition see for example [18] or [9]. We have the following theorem

Theorem 2.1 ([17]). *If C and C' are universal computing machines then*

$$\left| AIC(s, C) - AIC(s, C') \right| \le K\left(C, C'\right)$$

where $K\left(C, C'\right)$ is a constant which depends only on C and C'.

This theorem implies that the information content AIC of s with respect to C depends only on s up to a fixed constant, then its asymptotic behaviour does not depend on the choice of C. For this reason from now on we will write $AIC(s)$ instead of $AIC(s, C)$.

The shortest program which gives a string as its output is a sort of encoding of the string, and the information which is necessary to reconstruct the string is contained in the program. From this point of view the computer can also be seen as a decoder. Unfortunately the coding procedure associated to the Algorithmic Information Content cannot be performed by any algorithm. This is a very deep statement and, in some sense, it is equivalent to the Turing halting problem or to the Gödel incompleteness theorem. Then the Algorithmic Information Content is a function not computable by any algorithm. Hence in computations one tries to approximate from above the AIC of a string by means of some algorithm. This leads to consider the theory of *compression algorithms*, algorithms that encode an original string $s \in \mathcal{A}^*$ into a compressed version of it $M(s) \in \{0,1\}^*$ in a reversible way (i.e., there is an other algorithm that is able to recover the original string from the coded string). Using a compression algorithm M one defines the information content of the string s with respect to the encoding procedure M as $I_M(s) = |M(s)|$, where $|\cdot|$ denotes the binary length of $M(s)$. We remark that the Algorithmic Information Content of a string is up to a constant less than or equal to the information content as it is computed by some compression algorithm. That is, for each compression algorithm M there is a constant C_M such that for each s it holds $I_M(s) \geq AIC(s) + C_M$. This is because each universal computing machine can be programmed also to perform any coding-decoding technique, and the length of this program represents C_M.

To study a chaotic dynamical system we have to consider the asymptotic behaviour of its orbits, hence if we want to consider in some sense the information contained in the orbits of the system, we have to deal with infinite strings. Then let $\Omega := \mathcal{A}^{\mathbb{N}}$ denote the set of infinite strings ω with letters from $\mathcal{A}$. To study the information contained in a infinite string ω, we study what is the asymptotic behaviour of the information contained in the first n symbols of ω as n increases. Hence we study the asymptotic behaviour of the function $AIC(\omega, n) := AIC(\omega^n)$ as n increases, where $\omega^n = (\omega_0, \ldots, \omega_{n-1})$ is the string given by the first n symbols of ω. By the methods of symbolic dynamics we will associate an infinite string to an orbit and then the above idea will be applied to dynamical systems.

2.1 Application to dynamical systems

Let $(X, \mathcal{B}, T)$ be a dynamical system. X is assumed to be a compact metric space, $\mathcal{B}$ is the Borel σ-algebra, and T is a $\mathcal{B}$-measurable map from X to itself. Let μ be a T-invariant measure on $(X, \mathcal{B})$. We do not suppose that

$\mu(X) < \infty$, but we always assume μ to be σ-finite and conservative[1].

Let $Z = \{I_0, \ldots, I_{N-1}\}$ be a finite measurable partition of X, and let $\mathcal{A} = \{0, \ldots, N-1\}$ be the associated finite alphabet. Then the symbolic representation of the system (X, T, μ) is given by the function $\varphi_Z : X \to \Omega = \mathcal{A}^{\mathbb{N}}$ associating to each point a symbolic orbit with respect to Z, defined by

$$\varphi_Z(x) = \omega \iff T^i(x) \in I_{\omega_i} \quad \forall \, i \in \mathbb{N} \tag{3}$$

The image $\varphi_Z(X)$ is a subset of Ω which is invariant under the usual shift map τ on Ω. The function φ_Z induces on $\varphi_Z(X)$ a measure corresponding to μ. Of course similar symbolic representations can also be defined for countable partitions (with a countable symbolic alphabet).

At this point it is possible to apply the notion of information to the orbits of the system. We obtain quantities dependent on a given partition Z of X. The following definition and the following theorem, given in [8], shows the relation between AIC and entropy.

Definition 2.2. The *information content* $AIC(x, n, Z)$ of n steps of the orbit of x with respect to Z is defined as $AIC(x, n, Z) = AIC((\varphi_Z(x))^n)$. Analogously, the *complexity* $K(x, Z)$ of a point $x \in X$ with respect to Z is given by

$$K(x, Z) = \limsup_{n \to \infty} \frac{AIC(x, n, Z)}{n}. \tag{4}$$

The complexity of a sequence is related to the Shannon entropy of the information source that has produced the sequence. Hence in the theory of dynamical systems it is possible to relate the complexity to the Kolmogorov-Sinai entropy of the system. The following theorem holds

Theorem 2.3 ([8],[24]). *Let (X, T) be a dynamical system and μ a T-invariant probability measure on X. Given a finite measurable partition Z of X, it holds*

$$\int_X K(x, Z) \, d\mu(x) = h_\mu(T, Z)$$

where $h_\mu(T, Z)$ denotes the Kolmogorov-Sinai entropy of the system relative to the partition Z. If the dynamical system (X, T, μ) is ergodic then for μ-almost all $x \in X$

$$K(x, Z) = \liminf_{n \to \infty} \frac{AIC(x, n, Z)}{n} = h_\mu(T, Z) \tag{5}$$

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[1]A system is conservative if the set of wandering points has zero measure (see [1]).

In systems with an infinite measure we have a behaviour of the information that is similar to zero entropy systems, indeed the following holds.

Theorem 2.4 ([6]). *Let (X, T) be a dynamical system and μ a T-invariant infinite ergodic measure. Given a finite measurable partition Z of X, for μ-almost all $x \in X$ it holds $K(x, Z) = 0$.*

These theorems tell us what we can expect for the asymptotic behaviour of the information content of a typical orbit of an ergodic dynamical system. If the system has an invariant probability measure μ with positive Kolmogorov-Sinai entropy relative to a partition Z, then for μ-almost all $x \in X$ it holds[2]

$$AIC(x, n, Z) \sim n \, h_\mu(T, Z) \tag{6}$$

If instead the ergodic measure μ has null Kolmogorov-Sinai entropy or it is an infinite measure, then for μ-almost all $x \in X$

$$AIC(x, n, Z) = o(n) \tag{7}$$

for any finite partition Z. In this second situation we introduce the notions of *local* and *global chaos indexes* to classify dynamical systems according to the asymptotic behaviour of $AIC(x, n, Z)$.

Let (X, T) be a dynamical system and μ a T-invariant measure. Let ν be a probability measure on X equivalent to μ (each one is absolutely continuous with respect to the other). We refer to ν as a "reference measure". Of course when μ is a probability measure, we can set $\nu = \mu$. Let Z be a finite partition of X.

Definition 2.5. The *upper chaos index* $\overline{q}(T, Z, \nu)$ *with respect to Z is given by*

$$\overline{q}(T, Z, \nu) = \inf \left\{ q \in (0, 1) \; / \; \limsup_{n \to \infty} \int_X \frac{AIC(x, n, Z)}{n^q} \, d\nu(x) = 0 \right\} =$$

$$= \sup \left\{ q \in (0, 1) \; / \; \limsup_{n \to \infty} \int_X \frac{AIC(x, n, Z)}{n^q} \, d\nu(\omega) = \infty \right\}$$

In the same way the *lower chaos index* $\underline{q}(T, Z, \nu)$ *with respect to Z is defined using the inferior limit instead of the superior limit.*

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[2]NOTATIONS: Here and in the sequel, for two sequences a_n and b_n we shall write $a_n \sim b_n$ if the quotient a_n/b_n tends to unity as $n \to \infty$. Moreover, the notation $a_n \approx b_n$ means that $a_n/b_n = O(1)$ as well as $b_n/a_n = O(1)$ for $n \to \infty$. Moreover we shall write $a_n \preceq b_n$ if $a_n = O(b_n)$.

We remark that in principle the chaos index may depend on the choice of ν. In principle, even if we consider equivalent measures, the index may change. However we will see that in the class of map we are interested to study this does not happen. Moreover we remark that in the examples we study there is a natural choice of the reference measure, the Lebesgue measure.

We also define the *upper* and *lower local chaos indexes*.

Definition 2.6. The *upper local chaos index* $\overline{q}(T, x, Z)$ is defined as

$$\overline{q}(T, x, Z) = \inf \left\{ q \in (0, 1) \;/\; \limsup_{n \to \infty} \frac{AIC(x, n, Z)}{n^q} = 0 \right\} =$$

$$= \sup \left\{ q \in (0, 1) \;/\; \limsup_{n \to \infty} \frac{AIC(x, n, Z)}{n^q} = \infty \right\}$$

In the same way the *lower local chaos index* $\underline{q}(T, x, Z)$ is defined using the inferior limit instead of the superior limit.

Theorem 2.7 ([6]). *Let μ be ergodic and invariant for the dynamical system (X, T) and let ν be a reference measure. For any finite partition Z of X, the local indexes are a.e.-constant, that is for μ-almost all $x \in X$ it holds $\overline{q}(T, x, Z) = \overline{q}(T, Z)$ and $\underline{q}(T, x, Z) = \underline{q}(T, Z)$. Moreover*

$$\underline{q}(T, Z) \le \underline{q}(T, Z, \nu) \le \overline{q}(T, Z, \nu)$$

Now we want to get rid of the dependence on the partition Z and define an indicator that is independent on the partition. We will state a definition for maps over the interval. Further generalizations are possible (see [4]) but for the sake of simplicity here we will restrict to interval maps.

To avoid pathologies coming from very complicated partitions (see the "negative results" in [5]) we will consider a class of admissible partitions and take the supremum over this class.

Definition 2.8. A partition $Z = \{I_0, ..., I_{N-1}\}$ of $[0, 1]$ is called *admissible* if it is made of a finite set of intervals, i.e. each I_i is an interval. The *upper* and *lower global weak chaos indexes* of $([0, 1], T, \nu)$ are defined by

$$\overline{q}(T, \nu) = \sup_{Z \in \ adm. \ part.} \overline{q}(T, Z, \nu)$$

$$\underline{q}(T, \nu) = \sup_{Z \in \ adm. \ part.} \underline{q}(T, Z, \nu)$$

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What we obtained is a weak chaos index for maps of the interval that is in general invariant for a bi-Lipschitz conjugacy. Let us consider the Lebesgue measure m on the unit interval.

Theorem 2.9. *If $([0,1], T)$ and $([0,1], T')$ are conjugated by a bi-Lipschitz homeomorphism π, then $\overline{q}(T, m) = \overline{q}(T', m), \underline{q}(T, m) = \underline{q}(T', m)$.*

Proof. It is clear that a homeomorphism sends an admissible partition Z to an admissible partition $\pi(Z)$. We have that

$$\int_{[0,1]} \frac{AIC(\pi(x), n, \pi(Z))}{n^q} \, dx = \int_{[0,1]} \frac{AIC(x, n, Z)}{n^q} \, f(x) \, dx$$

where $f(x) \in L^\infty[0,1]$ and then we can estimate one index in function of the other. $\square$

3 The "Manneville-Pomeau like" maps

We apply our techniques, based on the asymptotic behaviour of the information content of symbolic orbits, to a family of interval differentiable maps $T : [0,1] \rightarrow [0,1]$ with an indifferent fixed point.

Definition 3.1. We say that a map $T : [0,1] \rightarrow [0,1]$ is a *Manneville-Pomeau map (MP map)* with exponent z if it satisfies the following conditions:

1. there is $c \in (0,1)$ such that, if $I_0 = [0,c]$ and $I_1 = (c,1]$, then $T\big|_{(0,c)}$ and $T\big|_{(c,1)}$ extend to C^1 diffeomorphisms, $T(I_0) = [0,1]$, $T(I_1) = (0,1]$ and $T(0) = 0$;

2. there is $\lambda > 1$ such that $T' \geq \lambda$ on I_1, whereas $T' > 1$ on $(0,c]$ and $T'(0) = 1$;

3. the map T has the following behaviour when $x \rightarrow 0^+$

$$T(x) = x + rx^z + o(x^z)$$

for some constant $r > 0$ and $z > 1$ (see left part of Figure 1).

This family of maps is well known and many statistical (the decay of correlations, the central limit theorem and the phenomenon of phase transitions) and ergodic (exactness, rational ergodicity, mixing and the return time sequences) properties have been deeply analyzed. Most of these studies

are made for $z < 2$, on the contrary we are mostly interested in the case $z \geq 2$.

For $z \in (1, 2)$ there is a unique absolutely continuous (with respect to Lebesgue measure) probability measure μ that is T-invariant, moreover μ is a Sinai-Ruelle-Bowen measure, it is exact and its Kolmogorov-Sinai entropy satisfies

$$h_\mu(T) = \int_0^1 \log |T'(x)| \, d\mu(x) > 0$$

From the statistical point of view, it is known that some changes happen when z crosses the value $z = \frac{3}{2}$. However from our point of view these changes are not relevant.

Applying Theorem 2.3 and the consequent equation (6) we obtain that if T is a Manneville-Pomeau map with $z \in (1, 2)$, and Z is the generating partition $Z = \{I_0, I_1\}$, then for the absolutely continuous T-invariant probability measure μ it holds

$$AIC(x, n, Z) \sim n \, h_\mu(T)$$

for μ-almost all $x \in [0, 1]$.

Much more delicate is to study the case $z \geq 2$. Indeed for these values of the parameter the only absolutely continuous T-invariant measure μ is infinite. Hence in this case we obtain from Theorem 2.4 and equation (7) that $AIC(x, n, Z) = o(n)$ for μ-almost all $x \in [0, 1]$, for any finite partition Z. To classify these maps we study the behaviour of the chaos indexes introduced in Section 2.

The ergodic properties of the infinite measure μ for MP maps with $z \geq 2$ have been studied in [20] and [21], applying the theory of infinite ergodic measures (see [1]). In particular the measure μ is shown to be exact and rationally ergodic, with estimates for the return time sequences. Putting together the results of [21] and the Darling-Kac Theorem ([1]) we obtain

Theorem 3.2 (Aaronson-Darling-Kac-Thaler). *Let T be a Manneville-Pomeau map with $z \geq 2$ and μ the infinite absolutely continuous T-invariant measure. Then for all Borel measurable $B \subset [0, 1]$ with $\mu(B) < \infty$ it holds*

$$\frac{1}{a_n} \sum_{j=0}^{n-1} \chi_B \circ T^j \overset{\mathcal{L}}{\Longrightarrow} \mu(B) \, Y_\alpha$$

where the convergence is in distribution with respect to all absolutely continuous Borel probability measures on $[0, 1]$, Y_α is a positive random variable distributed according to the normalized Mittag-Leffler law of order α and $\alpha = \frac{1}{z-1}$. Moreover it holds

- $a_n \sim \frac{n}{\log n}$ *if* $z = 2$;

- $a_n \sim n^{\frac{1}{z-1}}$ *if* $z > 2$.

The statistical distribution of the frequencies of visits to subsets of $[0, 1]$ is the fundamental tool to obtain the behaviour of the AIC of orbits of the system.

Our plan is the following:

(i) to consider a symbolic representation of the system induced by the choice of a partition;

(ii) to estimate the information content of the orbits using a particular encoding as a compression algorithm;

(iii) to show that this information content has the same average asymptotic behaviour as the Algorithmic Information Content.

To obtain a symbolic representation of the system we will consider a finite admissible partition (see Definition 2.8). First we will consider the case of a partition made of two intervals $I_0 = [0, c]$ and $I_1 = (c, 1]$, then we will show that the general case is similar. In this first particular case, the alphabet associated to the partition Z is $\mathcal{A} = \{0, 1\}$.

Let us consider a MP map. As said before the presence of the indifferent fixed point at the origin implies that a typical orbit will spend much time near the origin, since it moves away from it very slowly, and will spend the rest of time around in the interval. Then at some time it will come again close to the origin and again stay near the origin a lot of time. This repeats over and over again. This fact implies that a typical symbolic orbit will have a lot of symbols equal to "0" and some equal to "1". Moreover being the derivative of the map bounded from 1 in I_1, the dynamics in this interval is "fully" chaotic, and this implies a sort of renewal process when the orbit of the point is in I_1 (see Section 4). The results we will see are in some sense reminiscent of the theory of renewal processes, but this theory fails for MP maps, since they are not isomorphic to a Markov chain.

We now introduce the encoding we use to estimate the information. A typical sequence $\omega = \varphi_Z(x) \in \{0, 1\}^{\mathbb{N}}$ will look like[3]

$$\omega = (0001000000001100100000000001101\dots)$$

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[3]This symbolic representation is well defined for m-a.e. point $x \in [0, 1]$ (analogously to the case of the dyadic numbers for the Bernoulli shift).

so it is possible to compress its first 30 symbols in the following string

$$s(\omega^{30}) = (3802901) \tag{8}$$

where we have just written how many "0"s there are between two consecutive "1"s. Since the number of consecutive zeros can be as high as we want, the string s is a string with digits coming from an infinite alphabet (each digit is a natural number). Since we want to deal with strings coming from a finite alphabet we codify s into a binary string s', simply using the standard binary representation of natural numbers and writing it with the usual prefix-code (see below for an example). In this way a number $n \in \mathbb{N}$ is encoded by a binary string $(a_0, a_1, \ldots, a_m)$ such that

$$n = 2^{m+1} - 1 + \sum_{j=0}^{m} a_j 2^j$$

where $m + 1 = \lfloor \log_2(n + 1) \rfloor$ (denoting $\lfloor \cdot \rfloor$ the inferior integral part of a number), and this binary string is written using the prefix-code given by

$$\bar{n} = (1^{m+1}, 0, a_0, \ldots, a_m)$$

where 1^{m+1} denotes the symbol "1" repeated $m + 1$ times. In this way, leaving unchanged the symbols "0", we obtain for our example

$$s'(\omega^{30}) = (11000\ 1110001\ 0\ 101\ 1110010\ 100)$$

Hence $s'(\omega^n)$ is an encoding of the string ω^n (see Section 2).

The information function I associated to this compression is given by

$$I_{s'}(\omega^n) = |s'(\omega^n)| = \sum_{j=0}^{|s(\omega^n)|-1} 2 \lfloor \log_2(1 + s(\omega^n)_j) \rfloor + 1 \tag{9}$$

where $|\cdot|$ denotes the length of the string.

Let $N_n : \Omega \to \mathbb{N}$ be the sequence of functions defined by

$$N_n(\omega) = \#\{i \mid \omega_i \neq 0, \quad i = 0, \ldots, n-1\} \tag{10}$$

that is the number of passages of the orbit outside the interval I_0 in the first n steps. Then it is easy to realize that in the case of a partition made by two intervals $N_n(\omega) = |s(\omega^n)|$. In this case we now prove a stronger relation between $I_{s'}(\omega^n)$ and $N_n(\omega)$.

Lemma 3.3. *For any sequence $\omega \in \{0,1\}^{\mathbb{N}}$ it holds*

$$N_n(\omega) + 2 \, \log_2(n - N_n(\omega) + 1) \le I_{s'}(\omega^n) \le N_n(\omega) + 2 \, N_n(\omega) \log_2\left(\frac{n}{N_n(\omega)}\right)$$

up to an additive constant given by the possibilities $\omega_0 = 1$ and $\omega_{n-1} = 0$, and by the presence of the inferior integral part in the definition of $I_{s'}(\omega^n)$.

Proof. It is enough to prove the lemma for $\omega_0 = 0$ and $\omega_{n-1} = 1$. Otherwise simply add a constant.

In general $N_n = n - h$, for some $h < n$. The compression of such strings is then N_n-symbols long. Moreover the compression is such that $\sum_{j=0}^{N_n-1} s(\omega^n)_j = h$. We now want to find the maximum and the minimum of the function

$$\sum_{j=0}^{N_n-1} 2 \, \log_2(1 + s(\omega^n)_j)$$

with the condition $\sum_{j=0}^{N_n-1} s(\omega^n)_j = h$. The maximum is attained for equal $s(\omega^n)_j \ne 0$, and the minimum for all the $s(\omega^n)_j = 0$ but one which is equal to h. Then the maximum is given by $s(\omega^n)_j = \frac{h}{n-h}$ for all j, and the information content is given by $\sum_{j=0}^{n-h-1} \left[1 + 2 \, \log_2\left(1 + \frac{h}{n-h}\right)\right] = N_n + 2 \, N_n \log_2\left(\frac{n}{N_n}\right)$, and the minimum is given by $(n-h-1)+1+2 \, \log_2(h+1) = N_n + 2 \, \log_2(n - N_n + 1)$. Hence the lemma is proved. $\square$

Let us see what can be done when we have some general admissible partition made of a finite number of intervals. Let $Z = \{I_0, \ldots, I_{N-1}\}$ be such a partition and let $\mathcal{A}$ be the associated finite alphabet. In this case we assume $\mathcal{A}$ to be made of the symbol "0" for the interval I_0, and of letters (or any other kind of symbols different from natural numbers) for the other intervals.

We slightly modify the previous coding procedure as follows. We have strings $\omega^n \in \mathcal{A}^n$, and define $s(\omega^n) = (x_1, x_2, \ldots, x_m)$, where $x_i \in \mathbb{N} \cup \mathcal{A}$. In the sense that we codify as numbers the occurrences of the symbol "0" as before, in a way that if at place i there are n_i consecutive "0"s we write the number n_i. The other symbols are left unchanged. For example, if $\mathcal{A} = \{0, A, B\}$ and $\omega^n = (00ABB000A)$ then $s(\omega^n) = (2ABB3A)$. Then we define s' as before, by the standard binary encoding of the natural numbers, obtaining a string s' written in the alphabet $\{0, 1, A, B\}$. In the previous example, we obtain $s'(\omega^n) = (101 \, ABB \, 11000 \, A)$.

Then we can easily estimate the information function $I(\omega^n)$ in this case, by noting that the only difference with the previous case of a partition with only two intervals is that now when the symbol in ω^n is different from "0" we have to explicitly specify it, so that in $s'(\omega^n)$ the symbols "0" are replaced by the explicit strings, that is "ABB" and "A" in our example. Since there are $N_n(\omega)$ such symbols different from "0" or "1" in $s'(\omega^n)$, we need at most $N_n(\omega)\log_2(\#\mathcal{A} - 1)$ bits more than in the previous case with a partition with only two intervals. Note that in the previous case $\log_2(\#\mathcal{A} - 1) = 0$, hence this new term vanishes. We can now state the result

Lemma 3.4. *Let $\omega^n \in \mathcal{A}^n$ and s' be the encoding as above, then it holds*

$$I_{s'}(\omega^n) \leq\ N_n(\omega) + 2\ N_n(\omega)\log_2\left(\frac{n}{N_n(\omega)}\right) + N_n(\omega)\log_2(\#\mathcal{A} - 1)$$

where $\#\mathcal{A}$ is the cardinality of $\mathcal{A}$.

This lemma implies that the behaviour of the information content is given by the asymptotic behaviour of the functions N_n. To estimate the behaviour of N_n first of all notice that if x is a point in $[0,1]$ such that $\varphi_Z(x) = \omega$, then

$$N_n(\omega) = \sum_{j=0}^{n-1} \chi_{[0,1]\setminus I_0}(T^j(x))$$

Then we can apply Theorem 3.2 to the sequence N_n with respect to any measure ν on Ω induced by an absolutely continuous probability measure on $[0,1]$. This gives the estimates

$$\mathbb{E}_\nu[N_n(\omega)] \sim \begin{cases} \frac{n}{\log n}, & \text{z=2;} \\ n^{\frac{1}{z-1}}, & \text{z} > 2. \end{cases} \tag{11}$$

obtained by the asymptotic behaviour of the sequence a_n in Theorem 3.2, where $\mathbb{E}_\nu[\cdot]$ denotes the mean with respect to the measure ν. By the above estimates we have

Proposition 3.5. *Let T be a Manneville-Pomeau map with parameter $z \geq 2$. Let Z be an admissible finite partition. Then for any probability measure ν on Ω induced by an absolutely continuous probability measure on $[0,1]$ through the symbolic representation φ_Z, it holds*

$$\mathbb{E}_\nu[AIC(\omega^n)] \preceq \mathbb{E}_\nu[I(\omega^n)] \preceq\ n, \qquad\qquad z=2$$

$$\mathbb{E}_\nu[AIC(\omega^n)] \preceq \mathbb{E}_\nu[I(\omega^n)] \preceq\ n^{\frac{1}{z-1}}\log(n), \quad z > 2.$$

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Proposition 3.5 gives an estimate from above for the asymptotic behaviour of $\mathbb{E}_\nu[AIC(\omega^n)]$. Now we want to calculate a lower estimate. For this we consider a map that is derived by the MP map (it is an induced map) and which gives a symbolic dynamics that is similar to the compressed string $s(\omega^n)$. Proving that such a map has positive entropy we prove (using Theorem 2.3) that no further drastic compression is possible.

Let us consider again the partition $Z = \{[0, c], (c, 1]\}$, we will give a lower estimate to $\underline{q}(T, Z, \nu)$. The induced version of a MP map is obtained studying the passages through I_1. For any $x \in [0, 1]$ let $\tau(x)$ be the *time of the first passage through* I_1, that is

$$\tau(x) = 1 + \min\ \{n \geq 0 \mid T^n(x) \in I_1\} \tag{12}$$

The level sets of the function $\tau(x)$ are a partition of $(0, 1]$ into intervals $(A_n)_{n \geq 1}$ defined as

$$A_n = \{x \in [0, 1] \mid \tau(x) = n\} \tag{13}$$

Using $\tau(x)$ we define the *induced map* $G : [0, 1] \to [0, 1]$ by

$$G(x) = T^{\tau(x)}(x) \tag{14}$$

hence $G|_{A_n}(x) = T^n(x)$ and $G(A_n) = (0, 1]$, finally we define $G(0) = 0$ (see right part of Figure 1).
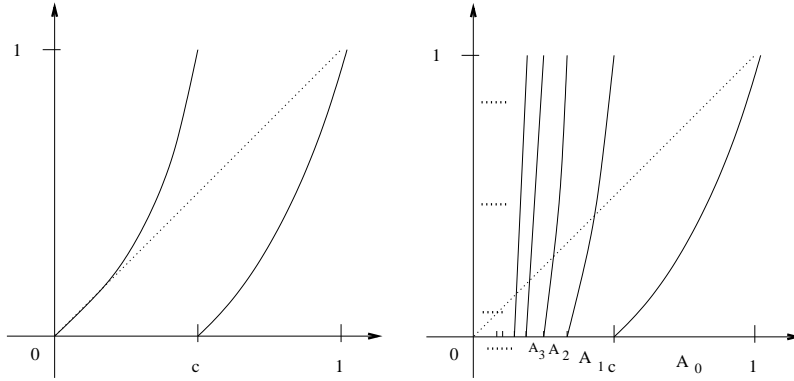

Figure 1: *An example of the graph of an MP map and its induced map.*

By the definition of the induced map, we have the following proposition.

Proposition 3.6. *Let T be a Manneville-Pomeau map and G its induced version on I_1. Let Z be the finite partition $Z = \{I_0, I_1\} = \{[0, c], (c, 1]\}$*

and P be the countable partition of $[0, 1]$ into the intervals $(A_n)_{n \geq 0}$, with $A_0 = \{[c, 1]\}$ and A_n for $n \geq 1$ defined as in equation (13). Then for a point x it holds $\varphi_Z(x) = \omega$ if and only if $\varphi_P(x) = s(\omega)^4$, where φ_P denotes the symbolic representation of the induced map G for the partition P.

The entropy of the induced map is positive. This is obtained putting together the results of Section 10 in [15] and Theorem 22 in [23].

Theorem 3.7 (Isola-Walters). *Let T be a Manneville-Pomeau map of parameter z and G be its induced version. Then there exists an absolutely continuous G-invariant probability measure ρ on $[0, 1]$ such that: (i) ρ is exact for G; (ii) $\rho(A_n) \sim n^{-1 - \frac{1}{z-1}}$, hence the entropy of the partition P respect to ρ is finite; (iii) $\infty > h_\rho(G) = \int_0^1 \log |G'| \, d\rho(x) > 0$.*

Proposition 3.8. *Let T be a Manneville-Pomeau map with exponent $z \geq 2$ and let Z be the finite partition $\{[0, c], (c, 1]\}$. Then using as reference measure any absolutely continuous probability measure ν on $[0, 1]$, it holds for the lower global weak chaos index*

$$\underline{q}(T, \nu) \geq \underline{q}(T, Z, \nu) \geq \begin{cases} 1, & z=2 \\ \frac{1}{z-1}, & z>2 \end{cases}$$

Proof. The inequality $\underline{q}(T, \nu) \geq \underline{q}(T, Z, \nu)$ follows trivially by definition. By Proposition 3.6 we have that if ω^n is a symbolic orbit of x with respect to Z then a symbolic orbit of the induced map G with respect to P is $s(\omega^n)$ and $|s(\omega^n)| = N_n(\omega)$. Given a string $s = (s_1 s_2 ... s_m)$ in $\mathbb{N}^*$ (like $s(\omega^n)$) let us consider the string $trunc_k(s) = (min(k, s_1) min(k, s_2) ... min(k, s_n))$ where each digit s_i in replaced by $min(k, s_i)$. Up to a constant, for each k $AIC(\omega^n) \geq AIC(trunc_k(s(\omega^n)))$ because the string $trunc_k(s(\omega^n))$ can be obtained easily from ω^n by an algorithm.

We know that the entropy of the induced map is finite and positive, then, since the infinite partition $P = \{A_0, A_1...\}$ is generating, its entropy $h_\rho(G, P)$ is finite and positive (Theorem 3.7 (iii)), then there is a partition P'_k of the form $P'_k = \{A_0, ..., A_k, \cup_{i>k} A_i\}$ such that $h_\rho(G, P) \geq h_\rho(G, P'_k) > 0$. This is because for each absolutely continuous measure the sequence of partitions P'_k converges to P in the Rokhlin metric (and the entropy is continuous with respect to change of partition in this metric).

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[4]By the definition of a compression algorithm, $s(\omega)$ makes sense only if ω is a finite string, hence this equation has to be intended valid for any finite substring ω^n of the sequence ω.

We remark that $trunc_k(s(\omega^n))$ is the symbolic orbit of x for the induced map, with respect to the partition P'_k and the length of this string is $N_n(\omega)$. Then by equation (5) of Theorem 2.3 and by Theorem 3.2, we have that given any sequence $\{b_n\}$ of positive numbers such that $b_n = o(a_n)$, where

$$a_n = \begin{cases} \frac{n}{log(n)}, & z=2 \\ n^{\frac{1}{z-1}}, & z>2 \end{cases}$$, and given any strictly positive constants h, C, ϵ such

that $h < h_\rho(G, P'_k)$, there exists an integer $\bar{n}$ such that for all $n \geq \bar{n}$ it holds $\rho(S(n)) > 1 - \epsilon$, where

$$S(n) = \{\omega \ / \ AIC(\omega^n) \geq AIC(trunc_k(s(\omega^n))) \geq hN_n(\omega) \ ; \ N_n(\omega) \geq Cb_n\}$$

Hence, being ρ absolutely continuous, for any absolutely continuous probability measure ν and for all $\delta > 0$ there exists a $\bar{n}$ (different from above) such that for each $n > \overline{n}$

$$\int_\Omega \frac{AIC(\omega^n)}{b_n} \ d\nu \ \geq \ hC \ \nu(S(n)) > hC(1 - \delta)$$

where h, C are as above. Hence for all sequences b_n such that $b_n = o(a_n)$ it holds

$$\liminf_{n \to \infty} \int_\Omega \frac{AIC(\omega^n)}{b_n} \ d\nu = \infty$$

hence the thesis follows from the definition of a_n. $\qquad \square$

We are finally ready to prove the main result of the paper.

Theorem 3.9. *If ν is an absolutely continuous probability measure, the global weak chaos indexes of a MP map with exponent $z \geq 2$ are given by*

$$\underline{q}(T, \nu) = \overline{q}(T, \nu) = \begin{cases} 1, & z=2 \\ \frac{1}{z-1}, & z>2 \end{cases}$$

Proof. From Proposition 3.5 we have the upper estimate

$$\overline{q}(T, Z, \nu) \leq \begin{cases} 1, & z=2 \\ \frac{1}{z-1}, & z>2 \end{cases}$$

for each admissible partition. Moreover from Proposition 3.8 we have the lower estimate for $\underline{q}(T, \nu)$. $\qquad \square$

It is well known that any two MP maps are topologically conjugated, the following shows that this conjugacy cannot be absolutely continuous if the exponents are different.

Corollary 3.10. If T_z and $T_{z'}$ are MP maps with different exponents $z \neq z'$ then there are not absolutely continuous conjugacies between T_z and $T_{z'}$.

Proof. As said before an homeomorphism sends an admissible partition Z to an admissible partition $\pi(Z)$. Considering the Lebesgue measure dx, we have that

$$\int_{[0,1]} \frac{AIC(\pi(x), n, \pi(Z))}{n^q} \, dx = \int_{[0,1]} \frac{AIC(x, n, Z)}{n^q} \, \pi(dx)$$

where $\pi(dx)$ is absolutely continuous. By Theorem 3.9, the global chaos index of the MP maps is the same for each absolutely continuous reference measure. By this and the above equation the chaos index of the systems $([0,1], T_z, dx)$, $([0,1], T_z, \pi(dx))$ and $([0,1], T_{z'}, dx)$ should be the same. Since the chaos index depends only on z, and $z \neq z'$ this leads to a contradiction. $\square$

We now conclude our exposition of results about the asymptotic behaviour of the Algorithmic Information Content of the MP maps with a result about the local behaviour. From Theorem 3.2 we can deduce that for almost each symbolic orbit ω of a MP map with respect to the partition $Z = \{[0, c], (c, 1]\}$ the following relations hold

$$\limsup_{n \to \infty} \frac{N_n(\omega)}{n^{\frac{1}{z-1}}} > 0 \quad \text{for} \quad z > 2, \tag{15}$$

$$\limsup_{n \to \infty} \frac{N_n(\omega)}{n \log(n)} > 0 \quad \text{for} \quad z = 2. \tag{16}$$

Hence applying the same techniques of the proof of Proposition 3.8 to the upper local chaos index, and recalling Theorem 2.7 we have

Proposition 3.11. If $Z = \{[0, c], (c, 1]\}$ is the generating partition with two intervals of a MP map, then for Lebesgue almost each $x \in [0, 1]$ it holds

$$\overline{q}(T, x, Z) = \overline{q}(T, Z) \geq \begin{cases} 1, & z=2 \\ \frac{1}{z-1}, & z>2 \end{cases}$$

This result has been applied in [7] to obtain quantitative recurrence results for the maps of the Manneville-Pomeau family.

4 Piecewise linear maps

We now study the behaviour of the Algorithmic Information Content for sequences obtained as symbolic representation of orbits of a class of piecewise linear maps whose properties are similar to the MP maps. In this case there is an isomorphism with a Markov chain with infinite states. We study the piecewise linear maps with the following properties.

Definition 4.1. Let $\{\epsilon_k\}_{k\in\mathbb{N}}$ be a sequence of positive real numbers, strictly monotonically decreasing and converging towards zero, with the property that

$$\frac{\epsilon_{k-1} - \epsilon_k}{\epsilon_{k-2} - \epsilon_{k-1}} < 1 \quad \forall\ k \in \mathbb{N} \tag{17}$$

We consider *piecewise linear (PL) maps* $L : [0, 1] \to [0, 1]$ defined by

$$L(x) = \begin{cases} \frac{\epsilon_{k-2} - \epsilon_{k-1}}{\epsilon_{k-1} - \epsilon_k}(x - \epsilon_k) + \epsilon_{k-1} & \epsilon_k < x \le \epsilon_{k-1}, \quad k \ge 1 \\ \frac{x - \epsilon_0}{1 - \epsilon_0} & \epsilon_0 < x \le 1 \\ 0 & x = 0 \end{cases} \tag{18}$$

where $\epsilon_{-1} = 1$. Clearly the properties of the maps depend on the sequence $\{\epsilon_k\}$, hence varying the sequence we obtain a class of PL maps.

Using the same approach of Section 3 we consider the partition $Z = \{I_0, I_1\}$, where $I_0 = [0, \epsilon_0]$ and $I_1 = (\epsilon_0, 1]$. Hence again we have the symbolic representation $\varphi_Z : [0, 1] \to \Omega = \{0, 1\}^*$. The general case of an admissible partition can be treated similarly as in Section 3. We also use the same information function used for the Manneville-Pomeau maps, hence the same kind of compression (see equation (8) and the definition of $s'(\omega^n)$), and study the information $I_{s'}(\omega^n) = |s'(\omega^n)|$ (see equation (9)).

Again Lemma 3.3 will apply, giving

$$\mathbb{E}_m[AIC(I(\omega^n))] \preceq \mathbb{E}_m[I(\omega^n)] \preceq \mathbb{E}_m[N_n] \log n \tag{19}$$

and then to have an upper bound to the information it is sufficient to study the behaviour of the random variables $N_n : \Omega \to \mathbb{N}$. We will see that the behaviour of N_n depends from the behaviour of (ϵ_k).

To obtain this we use the theory of infinite Markov chains. Indeed, due to their piecewise linearity, these maps are isomorphic to a Markov chain on an infinite alphabet, the natural numbers $\mathbb{N}$. The Markov chain is defined on the probability space $[0, 1]$ with the Lebesgue measure, and the chain is in the state i at time n if and only if $L^n(x) \in (\epsilon_{i-1}, \epsilon_{i-2}]$.

The transition matrix will look as follows:

$$\begin{pmatrix} (\epsilon_{-1} - \epsilon_0) & (\epsilon_0 - \epsilon_1) & (\epsilon_1 - \epsilon_2) & \cdots & (\epsilon_{n-2} - \epsilon_{n-1}) & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \quad (20)$$

This construction is well-known and we refer to [14] for its definition. To study the behaviour of this Markov chain the classical theory of Markov chains can be applied giving the results that follows.

Let t_0 be the "mean recurrence time of the passage through the interval I_1", in our case

$$t_0 = \sum_{k=1}^{+\infty} k \ (\epsilon_{k-1} - \epsilon_{k-2}) \quad (21)$$

The first result is the existence of an invariant measure for the Markov chain associated to our dynamical system.

Theorem 4.2 ([10]). *There is a measure $\bar{p}$ invariant for the Markov chain. The probability of the event k is defined by $\bar{p}(k) = \sum_{n=0}^{+\infty} (\epsilon_{n+k-1} - \epsilon_{n+k-2})$. This measure is finite if and only if the mean recurrence time t_0 is finite.*

The following result is obtained using the theory of recurrent events ([11]) and of power series ([22]).

Theorem 4.3. *Let t_0 be as above. If $t_0 < \infty$ then $\mathbb{E}[N_n] \sim \frac{n}{t_0}$, if instead $t_0 = \infty$, then $\mathbb{E}[N_n]$ is an infinite of order less than n. Moreover, let $F(x) = \sum_{r=1}^{[x]} (\epsilon_{k-1} - \epsilon_{k-2})$. If*

$$F(x) \sim 1 - Ax^{-\alpha} \quad (22)$$

as $x \to \infty$, where A is a constant and $\alpha > 1$ then $t_0 < \infty$.
If $F(x)$ is as above and $0 < \alpha < 1$, then $t_0 = \infty$ and

$$\mathbb{E}[N_n] \sim \frac{\sin \alpha\pi}{A\alpha\pi} n^{\alpha} \quad (23)$$

moreover if $b_n = o(\mathbb{E}[N_n])$ the set $R(n) = \{\omega \ / \ N_n(\omega) \leq b_n\}$ is such that

$$m(R(n)) \to 0. \quad (24)$$

If $F(x) \sim 1 - \frac{1}{\log x}$ then $t_0 = \infty$ and

$$\mathbb{E}[N_n] \sim \log n. \quad (25)$$

Proof. The proof of the first statements is based on a characterization of the mean $\mathbb{E}[N_n]$. In [11], it is shown that $\mathbb{E}[N_n] = U_n - 1$, where $U_n = \sum_{i=0}^{n} u_i$, being u_k the probability of being in the state "1" at time k. Then, using Theorem 1 in [11], it follows that

$$\lim_{n \to +\infty} \frac{\mathbb{E}[N_n]}{n} = \lim_{n \to +\infty} \frac{U_n}{n} = \lim_{n \to +\infty} u_n = \frac{1}{t_0}$$

Then, if $t_0 < +\infty$, that is "1" is ergodic, then $\mathbb{E}[N_n]$ is linear on n. Whereas if $t_0 = +\infty$, that is "1" is a null state, then $\mathbb{E}[N_n] = o(n)$.

Equations (23) and (24) follow from [11], Theorems 10 and 7.

Equation (25) is obtained using the results in [22], p.242, applications of Tauberian theorems, and repeating the argument of Theorem 10 in [11]. $\square$

Theorem 4.3 and equation (19) give, as in the previous section, an upper estimate of the Algorithmic Information Content of the symbolic orbits of our PL maps.

The final step is to have a lower estimate. This will be done as before, by considering an induced map G of equation (14). In this case the level sets of the first passage time are exactly the intervals $A_n = (\epsilon_{n-1}, \epsilon_{n-2}]$. In this case the G-invariant probability measure on $[0, 1]$ is the Lebesgue measure m, hence $m(A_n) = (\epsilon_{n-1} - \epsilon_{n-2})$ for all $n \geq 1$. Since the induced map G is now isomorphic to a stochastic process of independent and identically distributed random variables with values on $S = \{1, 2, 3, \dots\}$, the AIC of sequences in Ω is equivalent to the information function I if and only if the entropy of $\{A_n\}_{n \geq 1}$ with respect to m is finite, that is if and only if

$$H := -\sum_{n \geq 1} m(A_n) \, \log(m(A_n)) \, < \infty \tag{26}$$

Theorem 4.4. *Let L be a PL map on $[0, 1]$ and Z be the finite partition $\{I_0, I_1\}$. Then if m denotes the Lebesgue measure and the sequence (ϵ_k) satisfies equation (22) it holds*

$$\mathbb{E}_m[N_n] \preceq \mathbb{E}_m[AIC(x, n, Z)] \preceq \mathbb{E}_m[N_n] \log n$$

Proof. The first inequality follows directly from equation (19). If (ϵ_k) satisfies equation (22) then equation (26) is satisfied. The proof of the second inequality now is very similar to the proof of Theorem 3.8, we only have to use equation (24) instead of Theorem 3.2. $\square$

We now apply the results of this section to some particular PL maps, showing the behaviour of the global chaos indexes and the other features

in term of the behaviour of the sequence ϵ_k. In the first column we specify the behaviour of ϵ_k, the related value of t_0 is in the second column, then we have respectively the expected value of N_n (by 4.3), the entropy H of the induced map, the expected behaviour of the AIC of the orbits (by 4.4), the common value of the upper and lower global chaos indexes. The expected values and the chaos indexes are with respect to the Lebesgue measure. In the last example equation (26) is not satisfied, nevertheless the result for the Algorithmic Information Content follows already from the upper estimate of equation (19).

ϵ_k behaviour	t_0	$\mathbb{E}[N_n]$	H	$\mathbb{E}[AIC(\omega^n)]$	q
$\epsilon_k \sim \frac{1}{a^k}$ $a > 1$	$< \infty$	$\sim \frac{n}{t_0}$	$< \infty$	$\approx n$	1
$\epsilon_k \sim \frac{1}{k^\alpha}$ $\alpha > 1$	$< \infty$	$\sim \frac{n}{t_0}$	$< \infty$	$\approx n$	1
$\epsilon_k \sim \frac{1}{k^\alpha}$ $\alpha < 1$	∞	$\approx n^\alpha$	$< \infty$	$n^\alpha \preceq \cdots \preceq n^\alpha \log(n)$	α
$\epsilon_k \sim \frac{1}{\log(k)}$	∞	$\sim \log n$	∞	$\preceq (\log n)^2$	0

5   Conclusions

In this paper we considered two classes of weakly chaotic maps of the interval with a neutrally unstable fixed point. We calculated the information with respect to a partition and showed that this gives an invariant to characterize different weakly chaotic dynamics.

   This kind of behaviour for dynamical systems has been largely studied in the last years, from many different points of view. The importance of our approach lies in the fact that whereas the results given here are theoretical, the idea to use compression algorithms to study and measure experimentally the kind of chaos in intermittent dynamical systems can be practically exploited. In [4], using a particular compression algorithm that is suitable for null entropy strings, we performed experiments on some examples of intermittent and weakly chaotic dynamical systems and obtained results that are close to the theoretical predictions. Moreover, the study of weak chaos by compression algorithms gives rise to new questions in data compression (the search for algorithms that are optimal compressing zero entropy strings).

   We end remarking that we focused our interest onto maps of the interval. A more general approach to define a weak chaos index that is suitable for maps on a general metric space X is to use open covers instead of partitions ([12],[8], see also the remarks at the end of section 4 in [4]). In this paper we chose to simplify the question by using admissible partitions, however

all the results given here for the partitions hold also for the open covers. In [12] there is some example of results of this kind for PL maps.

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